Mean-field convergence in relative entropy for multi-species models and fluctuations in the moderate regime

Alexandra Holzinger, University of Oxford

Workshop: New Perspectives in Nonlocal and Nonlinear PDEs

8th of July 2025

joint work with José A. Carrillo and Shuchen Guo (University of Oxford)
& Li Chen (University Mannheim) and Ansgar Jüngel (TU Wien)

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Structure of the Talk

I. Mean-field derivations of Partial Differential Equations

- General introduction and moderate interactions
- Diffusion-aggregation model

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- multi-species model with Riesz-type kernels (Coulomb)

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$\downarrow$

Main technique: Quantitative convergence in $\mathbb{P}$

$\downarrow$

III. Fluctuations for moderately interacting particles

- single species, sub-coulomb interaction kernels

$\rightarrow$ joint work with Li Chen and Ansgar Jüngel '24

Short introduction to mean-field limits

Mean-field derivations of PDEs

- △ Goal: **Microscopic particle approximation** of partial differential equations
  - Particles can be molecules, bacteria, birds, humans, neurons
  - Idea goes back to Boltzmann (late 19th century), Hilbert (beginning of 20th century), Kac (1950s)

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- Interacting particle system of size N modelled by a system of SDEs on $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_i^N(t) = \frac{1}{N}\sum_{k=1}^N \nabla V(X_i^N - X_k^N)\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}B_i(t) \\ X_i^N(0) = \zeta_i \qquad i = 1, ..., N \end{cases}$$

- Interaction is captured by a *mean-value* of all interactions ($\nabla V =$ interaction kernel)
- $X_i^N(t)$ denotes position of i-th particle

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  - Interaction is captured by a *mean-value* of all interactions (∇V = interaction kernel)
  - $X_i^N(t)$ denotes position of i-th particle
- △ Limit: Show that for $N \to \infty$ the particle dynamics can be captured by

$$\partial_t u = -\mathrm{div}(u(\nabla V * u)) + \sigma \Delta u \quad t > 0, \quad u(0) = u_0$$

  - Solution u models the density function of a typical particle (modelled by a SDE)
- △ Empirical measure: $\mu_N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)} \to u(t)$ in distribution

Moderate Models

- $\triangle$ Classical framework: Interaction kernel ∇V appears as convolution in PDE
- $\triangle$ Can we derivate also local PDEs (without integral operators)?

Moderate Models

- △ Classical framework: Interaction kernel ∇V appears as convolution in PDE
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- ▶ Change the strength of interaction to moderate interaction

Moderate Models

△ Classical framework: Interaction kernel ∇V appears as convolution in PDE

△ Can we derivate also local PDEs (without integral operators)?

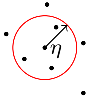
► Change the strength of interaction to moderate interaction

△ **Interaction kernel coupled to the number of particles** N in a certain way:

- New variable $\eta > 0$

$$V^\eta(x) := \eta^{-d} V(|x|/\eta) \to \kappa\delta_0 \quad \text{if } \eta \to 0$$

- If V has compact support on $B_1(0)$, then $\eta =$ *interaction radius*



△ Connect η and N (for example $\eta = N^{-\beta}$ or $\eta = \log(N)^{-\alpha}$) then for $N \to \infty$:

- Strength of interaction is stronger $\sim \frac{\eta^{-(d+1)}}{N} = N^{\beta(d+1)-1}$
- motivation for *moderate regime*

Moderate Models

△ How does the limiting PDE change?

△ Since $V^\eta \to \kappa\delta_0$ for $N \to \infty, \eta \to 0$ instead of

$$\partial_t u = -\mathrm{div}(u(\nabla V^\eta * u)) + \sigma\Delta u$$

we get

$$\to \partial_t u = -\kappa\mathrm{div}(u\nabla u) + \sigma\Delta u$$

- limiting equation is a local PDE

△ Oelschläger ('85)[1], Méléard and Roelly-Coppoletta ('87)[2]

[1] K. Oelschläger. A law of large numbers for moderately interacting diffusion processes, 1985.

[2] S. M and S. R. A propagation of chaos result for a system of particles with moderate interaction, 1987.

Strategy for moderate models

- Particle: $\mathrm{d}X_i^{N,\eta}(t) = \frac{1}{N}\sum_{k=1}^{N} \nabla V^\eta(X_i^N - X_k^N)\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}B_i(t)$
- Introduce an intermediate system of SDEs (uncoupled):

Intermediate level for $\eta > 0$ fixed

$$\mathrm{d}\bar{X}_i^\eta(t) = \left[\nabla V^\eta * u^\eta(t, \bar{X}_i^\eta)\right]\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}B_i(t)$$

where $\bar{X}_i(0) = \zeta_i$ and

$$\partial_t u^\eta = -\mathrm{div}(u^\eta(\nabla V^\eta * u^\eta)) + \sigma\Delta u^\eta$$

- Estimate $X_i^{N,\eta} - \bar{X}_i^\eta$ (like weak interaction)
- Non-local to local (singularity)

In which model are we interested in?

Diffusion-aggregation model

△ Consider a system of N interacting particles with positions $X_i^\eta(t) \in \mathbb{R}^d$

$$\begin{aligned} \mathrm{d}X_i^\eta(t) &= \frac{\kappa}{N}\sum_{j=1}^{N} \nabla V(X_i^\eta(t) - X_j^\eta(t))\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}W_i(t), \quad t > 0, \\ X_i^\eta(0) &= \zeta_i \quad \text{in } \mathbb{R}^d,\ i = 1, \dots, N. \end{aligned} \tag{1}$$

- $\kappa = 1$ (aggregating case), $\kappa = -1$ (repulsive case)
- $V = |x|^{-s}$ for $s > 0$, e.g. Keller-Segel, neural networks

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- $\kappa = 1$ (aggregating case), $\kappa = -1$ (repulsive case)
- $V = |x|^{-s}$ for $s > 0$, e.g. Keller-Segel, neural networks
  - well-defined? (especially for attractive systems difficult)

Diffusion-aggregation model

overcome: moderate interaction models!

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\end{aligned} \tag{1}
$$

- $\kappa = 1$ (aggregating case), $\kappa = -1$ (repulsive case)
- $V^\eta = \chi^\eta * \Phi, \quad \Phi(x) = |x|^{-s}$, where for $s > 0$

  $$\chi^\eta(x) = \eta^{-d}\chi(|x|/\eta)$$

  is a smooth mollification kernel

Moderate Regime for Riesz kernels - recent work

.. diffusion-aggregation system in the moderate regime:

$$\mathrm{d}X_i^\eta(t) = \frac{\kappa}{N}\sum_{j=1}^{N} \nabla V^\eta(X_i^\eta(t) - X_j^\eta(t))\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}W_i(t), \quad t > 0,$$

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Theorem (Olivera-Richard-Tomašević '21)

Let $0 < s < d-1$, then for $\eta = N^{-\alpha}$ for $\alpha > 0$ and $\mathfrak{p} > 0$ small enough

$$\mathbb{P}\Big(\|\chi^\eta * \mu_N - u\|_X > C\Big) \lesssim C^{-m}(C(u_0) + N^{-\mathfrak{p}+\varepsilon})^m \quad \forall m \in \mathbb{N}$$

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- $\|\cdot\|_X = \|\cdot\|_{L^\infty(0,T;L^1(\mathbb{R}^d)\cap L^r(\mathbb{R}^d))}$ for suitable $r > 1$,
- u solves

$$
\partial_t u = \sigma\Delta u - \kappa\text{div}(u\nabla\Phi * u),
$$

- first **quantitative propagation of chaos result** in this setting (allows for attractive regime)

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- first **quantitative propagation of chaos result** in this setting (allows for attractive regime)
- **However, too weak to study fluctuations around the mean-field limit**

Convergence in relative entropy for multi-species model and Newtonian singularity

joint work with José A. Carrillo and Shuchen Guo '25

Multi-species dynamics for Riesz kernels

- ... recent work with J. A. Carrillo, S. Guo, $\alpha = 1, \ldots, n$ (number of species)

$$\mathrm{d}X^{\eta}_{\alpha,i}(t) = -\frac{1}{N}\sum_{\beta=1}^{n}\sum_{j=1}^{N}\nabla V^{\eta}_{\alpha\beta}(X^{\eta}_{\alpha,i}(t) - X^{\eta}_{\beta,j}(t))\mathrm{d}t + \sqrt{2\sigma_\alpha}\mathrm{d}W_{\alpha,i}(t)$$

- limiting potential $V^{\eta}_{\alpha\beta} \to V_{\alpha\beta} \sim \pm|x|^{-s}$ up to Newtonian singularity $(0 < s \leq d-2)$

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- *strong* mean-field convergence

Theorem (Carrillo-Guo-H. '25)

Let $f_N = (f_{N,\alpha})$ be the particle distribution over all particles, under suitable condition on u_0

$$\sup_{0<t<T}\|f_N^{(K)} - \prod_{\alpha=1}^{n} u_\alpha^{\otimes K_\alpha}\|_{L^1(\mathbb{R}^{d|\mathbf{K}|})} \leq C(T)N^{-\zeta}$$

- **Parameters:** $0 < \zeta < 1 - \ell(2s+4)$, where and $\eta = N^{-\ell}$ with $0 < \ell < \frac{1}{2s+4}$ small
- **Limiting system:**

$$\partial_t u_\alpha = \sum_{\beta=1}^{n}\mathrm{div}(u_\alpha \nabla V_{\alpha\beta} * u_\beta) + \sigma_\alpha \Delta u_\alpha$$

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- Most recent work in relative entropy for moderate regimes: multi-species and **includes** Newtonian $\to$ **by path-wise estimates**

Relative Entropy for moderate models

...remember the idea for moderate regimes:

- introduce the intermediate level

$$\mathrm{d}\widetilde{X}^{\eta}_{\alpha,i}(t) = -\sum_{\beta=1}^{n} \nabla V^{\eta}_{\alpha\beta} * \widetilde{f}_{\beta,\eta}(t, \widetilde{X}^{\eta}_{\alpha,i}(t))\mathrm{d}t + \sqrt{2\sigma_\alpha}\mathrm{d}W_{\alpha,i}(t)$$

  where $\widetilde{f}_{\alpha,\eta}$ is the solution to the *mollified PDE*

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- split into I. **mean-field error** (to intermediate) and II. **PDE error** (from mollified PDE)

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- Using CKP-inequality $\|\mu - \nu\|_{L^1(E)} \leq \sqrt{2H(\mu|\nu)}$ with $H(\mu|\nu) = \int_E \mu \log \frac{\mu}{\nu}$

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- **Main step:** compare joint distribution of $X^{\eta}_{\alpha,i}$ to the one of $\widetilde{X}^{\eta}_{\alpha,i}$ denoted by $\widetilde{f}_N$

$$\sup_{t\in[0,T]} \frac{H\big(f_N(t)|\widetilde{f}_N(t)\big)}{N} \leq \frac{C(T)}{N^{1-\ell(2s+4)-\varrho}},$$

  where $\widetilde{X}^{\eta}_{\alpha,i}$ are independent $\to \widetilde{f}_N = \prod_{\alpha=1}^{n} \prod_{i=1}^{N} \widetilde{f}_{\alpha,\eta}$ and $\eta = N^{-\ell}$.

Relative Entropy for moderate models - crucial inequality

- f_N fulfils the Kolmogorov forward equation with $f_N(0) = \prod_{\alpha=1}^n (\bar{f}_\alpha^0)^{\otimes N}$:

$$\partial_t f_N = \sum_{\alpha,\beta=1}^n \sum_{i=1}^N \mathrm{div}_{x_{\alpha,i}} \big(f_N \frac{1}{N} \sum_{j=1}^N \nabla V_{\alpha\beta}^\eta (x_{\alpha,i} - x_{\beta,j}) \big) + \sum_{\alpha=1}^n \sum_{i=1}^N \sigma_\alpha \Delta_{x_{\alpha,i}} f_N,$$

- using the PDEs fulfilled by f_N, $\widetilde{f}_N$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{N} \mathcal{H}_N(f_N|\widetilde{f}_N)(t) + \sum_{\alpha=1}^n \sum_{i=1}^N \sigma_\alpha \int_{\mathbb{R}^{dnN}} f_N \Big| \nabla_{x_{\alpha,i}} \log \frac{f_N}{\widetilde{f}_N} \Big|^2 \mathrm{d}\mathbf{X}$$

$$\lesssim \mathbb{E}\Big[\frac{1}{N} \sum_{\alpha,\beta=1}^n \sum_{i=1}^N \Big| \nabla V_{\alpha\beta}^\eta * \widetilde{f}_{\beta,\eta}(X_{\alpha,i}^\eta) - \frac{1}{N} \sum_{j=1}^N \nabla V_{\alpha\beta}^\eta (X_{\alpha,i}^\eta - X_{\beta,j}^\eta) \Big|^2 \Big]$$

- Two additional difficulties:
  - Cross-interaction enters via terms like $\widetilde{f}_{\beta,\eta}(X_{\alpha,i}^\eta)$
  - We include $s = d - 2$ (Newtonian singularity)

How to estimate this term in the general settings?

- Pivoting argument

$$\mathbb{E}\left[\frac{1}{N}\sum_{\alpha,\beta=1}^{n}\sum_{i=1}^{N}\left|\nabla V_{\alpha\beta}^{\eta} * \widetilde{f}_{\beta,\eta}(X_{\alpha,i}^{\eta}) - \frac{1}{N}\sum_{j=1}^{N}\nabla V_{\alpha\beta}^{\eta}(X_{\alpha,i}^{\eta} - X_{\beta,j}^{\eta})\right|^{2}\right] \leq I_1 + I_2 + I_3,$$

where

- LLN terms involving terms like

$$\left|\nabla V_{\alpha\beta}^{\eta} * \widetilde{f}_{\beta,\eta}\big(\widetilde{X}_{\alpha,i}^{\eta}(t)\big) - \tfrac{1}{N}\textstyle\sum_{j=1}^{N}\nabla V_{\alpha\beta}^{\eta}\big(\widetilde{X}_{\alpha,i}^{\eta}(t) - \widetilde{X}_{\beta,j}^{\eta}(t)\big)\right|^{2} \checkmark$$

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- $$I_2 := \mathbb{E}\Big[\tfrac{1}{N}\textstyle\sum_{\alpha,\beta,i}\big|\tfrac{1}{N}\sum_{j=1}^{N}\nabla V^\eta_{\alpha\beta}\big(\widetilde{X}^\eta_{\alpha,i}(t) - \widetilde{X}^\eta_{\beta,j}(t)\big) - \tfrac{1}{N}\sum_{j=1}^{N}\nabla V^\eta_{\alpha\beta}\big(X^\eta_{\alpha,i}(t) - X^\eta_{\beta,j}(t)\big)\big|^2\Big]$$

- Basic Idea: Splitting $\Omega = C_\lambda \cup C_\lambda^c$
  1. $\omega \in C_\lambda$, **then** $\max_\alpha \max_{i=1,\dots,N} |X^\eta_{\alpha,i}(t,\omega) - \widetilde{X}^\eta_{\alpha,i}(t,\omega)| \le N^{-\lambda}$, MVT ✓
  2. the **complement**: then show that

$$\|\nabla V^\eta_{\alpha\beta}\|^2_{L^\infty}\mathbb{P}(C^c_\lambda) \le N^{-1/2-\varepsilon}$$

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- LLN terms involving terms like

$$\big|\nabla V^{\eta}_{\alpha\beta}*\widetilde{f}_{\beta,\eta}\big(\widetilde{X}^{\eta}_{\alpha,i}(t)\big)-\tfrac{1}{N}\textstyle\sum_{j=1}^{N}\nabla V^{\eta}_{\alpha\beta}\big(\widetilde{X}^{\eta}_{\alpha,i}(t)-\widetilde{X}^{\eta}_{\beta,j}(t)\big)\big|^{2}\ \checkmark$$

- $$I_2:=\mathbb{E}\Big[\tfrac{1}{N}\textstyle\sum_{\alpha,\beta,i}\big|\tfrac{1}{N}\sum_{j=1}^{N}\nabla V^{\eta}_{\alpha\beta}\big(\widetilde{X}^{\eta}_{\alpha,i}(t)-\widetilde{X}^{\eta}_{\beta,j}(t)\big)-\tfrac{1}{N}\sum_{j=1}^{N}\nabla V^{\eta}_{\alpha\beta}\big(X^{\eta}_{\alpha,i}(t)-X^{\eta}_{\beta,j}(t)\big)\big|^{2}\Big]$$

- Basic Idea: Splitting $\Omega=C_\lambda\cup C_\lambda^c$
  1. $\omega\in C_\lambda$, **then** $\max_\alpha\max_{i=1,\dots,N}|X^{\eta}_{\alpha,i}(t,\omega)-\widetilde{X}^{\eta}_{\alpha,i}(t,\omega)|\leq N^{-\lambda}$, MVT ✓
  2. the **complement**: then show that

$$\|\nabla V^{\eta}_{\alpha\beta}\|^2_{L^\infty}\mathbb{P}(C^c_\lambda)\leq N^{-1/2-\varepsilon}$$

…**convergence in** $\mathbb{P}$**:**

$$\mathbb{P}\Big(\max_{\alpha=1,\dots,n}\max_{i=1,\dots,N}|X^{\eta}_{\alpha,i}(t,\omega)-\widetilde{X}^{\eta}_{\alpha,i}(t,\omega)|>N^{-\lambda}\Big)\leq C(\gamma,T)N^{-\gamma}.$$

Convergence in $\mathbb{P}$ for moderate settings

Theorem (Mean-field in $\mathbb{P}$ for multi-species, Carrillo-Guo-H. 2025)

Let, $0 < s \le d-2$ and $\eta = N^{-\ell}$ where $0 < \ell < \frac{1}{2s+4}$ and $\ell < \lambda < \frac{1}{2} - \ell(s+1)$. Then it holds that for every $\gamma > 0$ there exists a $C(\gamma, T)$ such that

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- **Basic idea:** Stopping time argument + Gronwall's lemma for the quantity

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where $S^p_\lambda(t) = N^{\lambda p} \max_\alpha \max_i |X^\eta_{\alpha,i}(t,\omega) - \widetilde{X}^\eta_{\alpha,i}(t,\omega)|^p$ for

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- Reason for sub-coulomb setting $0 < s < d-2$ was **Taylor's expansion**

$$\| |D^2 V^\eta| * u^\eta(t) \|_{L^\infty(\mathbb{R}^d)} \leq C$$

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- definition of an auxiliary kernel K^η ($\sim$ cut-off Newtonian potential) and pivoting instead of Taylor to include $s = d-2$ in attractive case

Fluctuations around the mean-field limit

joint work with Li Chen and Ansgar Jüngel '24

Definition of the fluctuation process

- Recall: Mean-field setting:
  - Particle dynamics $\rightsquigarrow$ empirical measure $\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)}$
  - Mean-field result implies $\mu_N \to u$ (PDE solution) in distribution

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- △ Define $\mathcal{F}_N(t) := \sqrt{N}(\mu_N(t) - u(t))$

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- △ If $\mathcal{F}$ is a Gaussian process $\to$ **Central Limit Theorem**

Fluctuations in the moderate setting

△ Literature (few results)

- Results by Oelschläger '87 for repulsive particles
- Jourdain and Méléard '98, but not with $\sqrt{N}$ scaling
  - $\eta = \log(N)^{-\alpha}$ **not algebraic**
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△ Goal: Generalise result by Oelschläger such that it includes

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△ Main idea: **asymptotic fluctuation process**

$$\mathcal{F}_N := \sqrt{N}(\mu_N - u^\eta) \to \text{Gaussian} \quad u^\eta - u \to 0 \quad \text{for } N \to \infty$$

Remember: $\eta = N^{-\ell}$

Asymptotic CLT for aggregation of sub-coulomb type

Theorem (CLT, Chen-H.-Jüngel, 2024)

For $0 < s < d-1$, $\Phi(x) = |x|^{-s}$, and under suitable assumptions on ℓ, u_0, the conditional law fulfils

$$\mathrm{Law}(\langle \mathcal{F}_N(t), \phi\rangle | \mathscr{F}_0) \to \mathcal{N}\left(\langle \mathcal{F}_0, T^t_\phi(0)\rangle, 2\sigma \int_0^t \langle u(s), |\nabla T^t_\phi(s)|^2\rangle \mathrm{d}s\right).$$

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- The limiting dynamics $\to$ **non-local PDE solution with Riesz potential**, through $T^t_\phi(s)$, which solves

$$-\partial_s v - \sigma \Delta v = \kappa(\nabla\Phi * u)\cdot \nabla v - \kappa \nabla\Phi * (u\nabla v) \quad \text{in } \mathbb{R}^d,\ s \in (0,t), \quad v(t) = \phi,$$

where u solves

$$\partial_t u = \sigma\Delta u - \kappa \mathrm{div}(u\nabla\Phi * u), \quad t > 0, \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d$$

Main idea – in a nutshell

△ **Moderate Regime:** $\eta = N^{-\ell}$ with $0 < \ell < \frac{1}{8s+12}$ and $V^\eta = Z^\eta * Z^\eta$ and

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- taking $\mathbb{E}$ $\rightarrow$ same term as in relative entropy! (**use convergence in** $\mathbb{P}$)

Conclusion & Outlook

We showed:

I. Convergence in relative entropy for moderately interacting particles
   - strong notion of propagation of chaos
   - included **multi-species dynamics**
   - included **Coulomb setting**

II. Asymptotic Central Limit Theorem for attractive Riesz kernels (subcoulomb)
   - **main ingredient:** Convergence in Probability
   - generalized Oelschläger's proof in the **moderate regime**
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I would be interested in:

- include higher singularities $s > d - 2$
- the **original moderate regime**, i.e. $V^{\eta} \to \delta_0$ in the attractive case
- can we show it also in the weak regime (subcoulomb particle system is well-defined in attractive case)?
- can we show a **multi-species** fluctuation result?

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Thank you for your attention!